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Morphisms, Hemimorphisms, and Baer *-Semigroups

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The relationship between CROCs (complete orthomodular lattices) and complete Baer *-semigroups is discussed using an explicit construction of the adjoint of a hemimorphism. Simple examples provide much insight into the structures involved.

1. PRELIMINARIES

A CROC is nothing else than a complete orthomodular lattice (Piron, 1976). We call it a CROC as it is canonically relatively orthocomplemented, which means that each segment [0, a], that is, the set of elements between 0 and a, is by itself a complete orthomodular lattice, where the orthocomplementation is defined by $x^r = x' \wedge a$. A hemimorphism from a CROC $\mathcal{A}$ to a CROC $\mathcal{B}$ is a map ϕ from $\mathcal{A}$ to $\mathcal{B}$ which maps 0 to 0 and preserves the supremum:

$$\phi(\vee_i a_i) = \vee_i \phi(a_i)$$

According to the usual definitions, a hemimorphism which conserves the orthogonality relation is called a morphism.

A complete Baer *-semigroup is a set S equipped with the following (Foulis, 1960; see also Pool, 1968):

(i) An associative multiplication law with a (necessarily unique) 0 and I:

$$(fg)h = f(gh),$$
$$0f = f0 = 0 \qquad \forall f \in S$$
$$If = fI = f \qquad \forall f \in S$$

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(ii) An involution $f \mapsto f^*$:

$$f^{**} = f$$
$$(fg)^* = g^* f^*$$

in such a way that:

(iii) Each annihilator $\{f | fg = 0 \ \forall g \in M \subset S\}$ is an ideal of the form Sp, where p is a projection, that is, an element of S such that

$$p = p^* = p^2$$

One can easily show that 0 and I are projections and that the annihilator of 0 is generated by I and that of I by 0.

The aim of the next two sections is to show that these two structures are intimately linked. Finally we consider an instructive example in Section 4.

2. THE COMPLETE BAER *-SEMIGROUP ASSOCIATED TO A CROC

Let $\phi: \mathcal{A} \to \mathcal{B}$ and $\psi: \mathcal{B} \to \mathcal{A}$ be hemimorphisms. Then by definition ϕ and ψ form an adjoint pair if the following two conditions are satisfied:

$$\psi(\phi a)' < a' \qquad \forall a \in \mathcal{A}$$
$$\phi(\psi b)' < b' \qquad \forall b \in \mathcal{B}$$

Surprisingly, given any hemimorphism ϕ, there exists a hemimorphism ψ such that ϕ and ψ form an adjoint pair. More precisely we have the following result:

Lemma. Each hemimorphism $\phi: \mathcal{A} \to \mathcal{B}$ has a unique adjoint $\phi^*: \mathcal{B} \to \mathcal{A}$ given by

$$\phi^* b = \bigwedge_{\phi a < b'} a'$$

Proof. We first show unicity. Let ϕ^* and ϕ^+ be adjoint to ϕ and set $\phi^* b = a$. Then $\phi a' = \phi(\phi^* b)' < b'$ since ϕ and ϕ^* are adjoint. Hence $b < (\phi a')'$. But then, since ϕ^+ is monotone we have that $\phi^+ b < \phi^+(\phi a')' < a = \phi^* b$, where we have used the fact that ϕ and ϕ^+ are adjoint. Interchanging the roles of ϕ^* and ϕ^+, we have that $\phi^* = \phi^+$.

We now show existence. Define

$$\phi^* b = \bigwedge_{\phi a < b'} a$$

We first show that ϕ^* is a hemimorphism. It is trivial that $\phi^* 0 = 0$. Further,

$$\phi^*(\vee_i b_i) = \bigwedge_{\phi a < (\vee_i b_i)'} a = \bigwedge_{\phi a < b_i' \, \forall b_i} a = \vee_i \phi^*(b_i)$$

We now show that ϕ and ϕ^* form an adjoint pair. We have that

$$\phi^*(\phi a)' = \bigwedge_{\phi x < \phi a} x' < a'$$

by considering $x = a$. On the other hand,

$$\phi(\phi^* b)' = \phi(\bigwedge_{\phi a < b'} a')' = \phi(\bigvee_{\phi a < b'} a) = \bigvee_{\phi a < b'} \phi a < b'$$

where we have used the fact that ϕ preserves the supremum, completing the proof.

Example. Let J_a be the canonical injection J_a: $[0, a] \to \mathcal{A}$. We show that $J_a^* J_a = I$ on $[0, a]$ and that $J_a J_a^* = \phi_a$ on $\mathcal{A}$, where ϕ_a is the Sasaki projection defined by $\phi_a x = (x \vee a') \wedge a$. The projections are then self-adjoint and idempotent: $\phi_a = \phi_a^* = \phi_a^2$. Indeed, for $x \in \mathcal{A}$ and $y \in [0, a]$ we have that

$$J_a^* x = \bigwedge_{J_a y < x'} y^r = (\bigvee_{\substack{y < a \\ y < x'}} y)^r = (a \wedge x')^r = (x \vee a') \wedge a$$

and hence $J_a^* J_a y = J_a^* y = y$ (exactly the orthomodularity condition) and

$$J_a J_a^* x = (x \vee a') \wedge a = \phi_a x$$

Example. We show that a hemimorphism u is an isomorphism if and only if $u^* = u^{-1}$. Indeed, let u be an isomorphism, then $u(u^{-1}b)' = (uu^{-1}b)' = b'$ and $u^{-1}(ua)' = (u^{-1}ua)' = a'$, so that the two conditions on an adjoint pair are satisfied and $u^* = u^{-1}$. Conversely, let $u^{-1} = u^*$; then the first condition imposes that $u^{-1}(ua)' < a'$, which means that $(ua)' < ua'$. Furthermore, setting $b = (ua)'$ for any given a, the second condition imposes that $u(u^{-1}b)' < b'$, which means $(u^{-1}b)' < u^{-1}b'$, and by substitution, $(u^{-1}(ua))' < a$ and also $a' < u^{-1}(ua)'$ and so $ua' < (ua)'$. Hence $ua' = (ua)'$ and so u is an isomorphism.

Theorem. The set S of hemimorphisms of a CROC $\mathcal{A}$ into itself, equipped with the composition law and the adjoint defined above, forms a complete Baer *-semigroup.

Proof. (i) It is clear that the composition law of hemimorphisms is associative and that the hemimorphisms $a \mapsto 0$ and $a \mapsto a$ play the role of 0 and I, respectively.

(ii) The adjoint operation $\phi \mapsto \phi^*$ is well defined and $\phi^{**} = \phi$ since the conditions on an adjoint pair are symmetric. Finally, $(\psi\phi)^* = \phi^*\psi^*$ since the two conditions are satisfied. Indeed, from $\psi^*(\psi\phi a)' < (\phi a)'$ we derive the first, $\phi^*\psi^*(\psi\phi a)' < \phi^*(\phi a)' < a'$, and we obtain the second, $\psi\phi(\phi^*\psi^*b)' < b'$, by the same kind of reasoning.

(iii) Finally let $M \subset S$ and set $a_i = (\phi_i I)'$ for each $\phi_i \in M$. Define $a = \wedge_i a_i$. We show that the annihilator $\{\psi \mid \psi\phi_i = 0 \ \forall\phi_i \in M\}$ is identical to the ideal $S\phi_a$, where ϕ_a is the Sasaki projection $\phi_a x = (x \vee a') \wedge a$. Obviously $S\phi_a$ will be contained in the annihilator of M since $\phi_a\phi_i = 0$ for each $\phi_i \in M$. Indeed

$$\phi_a\phi_i x < \phi_a\phi_i I = \phi_a a_i' = (a_i' \vee a') \wedge a = 0$$

On the other hand, let ψ be in the annihilator of M, $\psi\phi_i = 0$ for all $\phi_i \in M$. We show that $\psi = \psi\phi_a$. Now $\psi(\phi_i I) = 0$, so that $\psi^* I = \psi^*(\psi(\phi_i I))' < (\phi_i I)' = a_i$ and so $\psi^* x < a_i$ for all a_i. Hence $(\phi_a\psi^*)x = \psi^* x$ for all x. This means that $\phi_a\psi^* = \psi^*$, and so $\psi = \psi\phi_a$ by taking the adjoint.

3. THE CROC ASSOCIATED TO A COMPLETE BAER *-SEMIGROUP

We can define a partial order relation on the set of projections of a complete Baer *-semigroup by setting $p < q$ if $p = pq$. This order relation can readily be seen to be identical to the set-theoretic inclusion

$$Sp \subset Sq$$

To each element $f \in S$ we will associate the projection f' which generates the annihilator of f:

$$\{g \mid gf = 0, g \in S\} = Sf'$$

Such a projection exists by definition and we have the following properties:

(i) If p is a given projection, then $p < p''$. Indeed, since in particular $p'p = 0$, then $(p'p)^* = pp' = 0$ and so p is in the annihilator of p'. This means that there exists an f with $p = fp'' = fp''p'' = pp''$.

(ii) If p and q are two projectors such that $p < q$, then $q' < p'$. Indeed by taking the adjoint of $p = pq$, we find that $p = qp$, which implies $q'p = q'(qp) = (q'q)p = 0$ and so $q' < p'$.

From these two properties the map $p \mapsto p''$ is a closure operation and $p' = p'''$. This justifies the following definition: p is a closed projector if $p = p''$. Note that 0 and I are closed since $0' = I$ and $I' = 0$.

Theorem. The set $\mathcal{A}$ of closed projectors of a complete Baer *-semigroup S, equipped with the partial order defined above and the orthogonality map $p \mapsto p'$, is a CROC.

Proof. (i) To show that $\mathcal{A}$ is a complete lattice it suffices to show that there exists a closed projector $\wedge_i p_i$, the infimum of any given family $\{p_i\}$, since there is a maximal element I. Now $p < q$ if and only if $Sp \subset Sq$ and so the infimum of a family $\{p_i\}$ must be associated to $\cap_i Sp_i$. However, as each p_i is closed, this is just the annihilator of the family $\{p_i'\}$, which is by definition generated by some projection p. It therefore remains to show that p is necessarily closed. Since $p < p_i$ we have that $p_i' < p'$ and so $p'' < p_i'' = p_i$ for all p_i. Hence $p'' < p$. On the other hand, $p < p''$, as p is a projection.

(ii) We now show that the map $p \mapsto p'$ is an orthocomplementation. We have that $p' = p'''$, so that the map is well defined. The map is trivially involutive and order reversing. Finally, $p \wedge p' = 0$, since if $q < p$ and $q < p'$, then $q = qp$ and $q = qp'$, giving $q = qp = qp'p = 0$.

(iii) The orthomodular law states that if $p < q$, then $(q' \vee p) \wedge q = p$. In fact it suffices to show that $(q \wedge p')' \wedge q < p$, since the opposite inequality is trivial; that is, we must show that $(q \wedge p')' \wedge q$ is in the annihilator of p'. We use the fact that in general if $pq = qp$, then $pq' = q'p$, $q \wedge p = pq$, and $q \wedge p' = qp'$. Indeed, let $pq = qp$. Then $(q'p)q = (q'q)p = 0$. Hence $q'p = q'pq'$ and so by taking the adjoint, $q'p = pq'$. Further, it is simple to show that if $pq = qp$, then pq is a projection, in fact the projection $p \wedge q$ (von Neumann, 1950, Theorem 13.4(1), p. 53). Now $pqq = pq$, so that $pq < q$ and in the same way $pq < p$. Finally, if $r < p$ and $r < q$, then $r = rp$ and $r = rq$, so that $r = rpq$ and $r < pq$. Now let $p < q$. Then $pq = qp$, so that $q \wedge p' = qp'$. Then $q(q \wedge p') = (q \wedge p')q$ and so $(q \wedge p')' \wedge q = (qp')'q$. But then it is trivial that $(qp')'qp' = 0$, completing the proof.

Theorem. Let $\mathcal{A}$ be a CROC and S the associated Baer *-semigroup. Then the CROC associated to S is exactly $\mathcal{A}$.

Proof. We need to show that the closed projections of S are exactly of the form ϕ_a for some $a \in \mathcal{A}$. We use the fact that $\phi' = \phi_a$ for $a = (\phi I)'$ as shown in Section 2 and so $(\phi_a)' = \phi_{a'}$. Each projection of the form ϕ_a is then closed since $(\phi_a)'' = (\phi_{a'})' = \phi_a$. On the other hand, let ϕ be a closed projection: $\phi = \phi''$. Then $\phi' = \phi_a$ for $a = (\phi I)'$ and so $\phi = \phi'' = (\phi_a)' = \phi_{a'}$, completing the proof.

Note that one cannot pass from a complete Baer *-semigroup to the associated CROC and back again in general. Indeed, let S be any field considered as a complete Baer *-semigroup under multiplication, where we take the identity as the involution. Then there are only two projections, namely 0 and I, since $a^2 = a$ implies $a(a - 1) = 0$. Hence all such S have the same associated trivial CROC $\{0, 1\}$. Note that this CROC has only two hemimorphisms as one can send 1 to either 0 or 1.

Table I.

$\phi_{\alpha\beta}$	$(\phi_{\alpha\beta})^*$	Self-adjoint	Idempotent	Closed	Morphism
00	00	Y	Y	Y	Y
$0a$	$0a$	Y	Y	Y	Y
$0a'$	$a0$	N	N	N	Y
01	aa	N	Y	N	Y
$a0$	$0a'$	N	N	N	Y
aa	01	N	Y	N	N
aa'	aa'	Y	N	N	Y
$a1$	$a1$	Y	N	N	N
$a'0$	$a'0$	Y	Y	Y	Y
$a'a$	$a'a$	Y	Y	Y	Y
$a'a'$	10	N	Y	N	N
$a'1$	$1a$	N	Y	N	N
10	$a'a'$	N	Y	N	Y
$1a$	$a'1$	N	Y	N	N
$1a'$	$1a'$	Y	N	N	N
11	11	Y	Y	N	N

4. AN EXAMPLE

In this final section we will consider the simplest nontrivial CROC which has four elements, namely 0, a, a', and 1. In this case there are 16 hemimorphisms ϕ, as one can send a and a' independently to an arbitrary element, and set $\phi 1 = \phi a \vee \phi a'$. This example, although very simple, exhibits much of the relevant structure of the set of hemimorphisms. For example, one of the hemimorphisms will be seen to be a projection which is not closed. Further, one can see that the adjoint of a morphism need not be a morphism.

The 16 hemimorphisms of will be labeled $\phi_{\alpha\beta}$, where α is the image of a' and β is the image of a. Hence, for example, the identity hemimorphism is $\phi_{a'a}$. Table I gives the adjoint of each hemimorphism and states whether a given hemimorphism is self-adjoint, idempotent, closed, or a morphism (Y, yes; N, no).

Hence we see that there are four closed projections, ϕ_{00}, ϕ_{0a}, $\phi_{a'0}$, and $\phi_{a'a}$, which give back the original CROC. There is one projection which is not closed, namely ϕ_{11}. Finally there are two morphisms whose adjoints are not morphisms, namely ϕ_{01} and ϕ_{10}.

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